• CLASSICS IN MATHEMATICS EDUCATION

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ALGEBRA AND THE DEVELOPMENT OF REASON*

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EDITOR'S NOTE:—Although the language and symbolism of this article may appear a bit archaic, the ideas expressed seem quite consistent with current emphases in mathematics education. This article was chosen to exemplify the position taken by Professor Keyser in the paper recently reprinted in this department.—F. JOE CROSSWHITE.

WHAT I wish to say to you today is a general reflection on the subject of high school algebra. As a layman both in mathematics and in secondary school teaching, I can speak only from two lay points of view—that of the pupil, which meets you at one end of your activity, and that of the philosopher, which you encounter at the other. Consequently, I shall begin by talking about the futility and barrenness of algebra, and end, I hope, by reviewing with you its importance, interest, and charm. For it is a peculiarity of the subject that an uninitiate mind can usually see nothing in it but a dry, lifeless discipline, whereas the adept sees in it the apotheosis of human reason.

All sorts of systematic and skillful methods have been devised for teaching young people algebra, and the amount of technique that can be imparted in a single year to perfectly immature minds is a constant source of wonder to me. But another and less agreeable mystery lies in the fact that so much algebra can impinge upon the immature mind without helping in the least to mature it. The subject has, apparently, no humanistic value—"humanistic" meaning beneficial to the development of mental power. Children who have learned algebra are not better thinkers than those who have not learned it. They have acquired a new technique, a technique of something strange and abstract; but neither they nor we can tell just why they have learned it. Its practical value is very limited: engineers, surveyors, and astronomers use it. Perhaps two to three percent of our students will be engineers, surveyors, astronomers, or members of some other profession that actually requires the technique of algebra; why must the other ninety-odd percent of every class be drilled in this esoteric art? It is useless, formal, and difficult; at best, it has the virtues of chess—it is a sophisticated game, known to sharpen the wits. We do not require a satisfactory chess record for graduation. Why did we ever introduce the complicated game of algebra—less useful than arithmetic, which it otherwise closely resembles, less lucid than geometry, to which it is somewhat mysteriously related?

Algebra was originally included in the school curriculum for a thoroughly humanistic purpose: it is the only elementary discipline which furnishes material for abstract thinking, and caters to the development and pleasure of the faculty called pure reason. In arithmetic, the elementary operations can be visualized as patterns arising from the combination of various

* The original article appeared in the May 1931 issue of this journal (Vol. XXIV, No. 5, pp. 285–97).

groups—we can visualize $4 + 5$ as two related groups of four and five objects respectively, or 4×5 as four examples of the latter grouped together—so that we may literally *see* the results of arithmetic, at least in the early stages of learning; sensuous intutition feels fairly at home with number-work. But in algebra we cannot correlate our quantities with rhythmic or configured patterns; a and b have not the definite individuality of 3 and 4. Their relations are not open to inspection by the mind's eye, naïvely and instantaneously; they must be understood through the *discursive reason*. It is for the sake of this kind of reasoning that algebra was originally taught in our schools; and this, in my opinion, is its only excuse for being in the curriculum. All its other purposes—its relation to special sciences, its traditional dignity, and so forth—seem to me farfetched in relation to the public, or else trivial. As a foundation for a few specific sciences, it certainly is not needed by literary-minded girls and boys, by future businessmen, lawyers, librarians, Latin teachers, kindergartners, or any number of other professions that require at least a high school education. As a mere demand of the colleges it is silly. But as a gymnastic in pure unaided reasoning, it is not only important, but unique. In that capacity, it concerns not only the scientist who will operate with its formulae, but every mind that is initiated by its subtleties into a new realm, the realm of abstract thought. For algebra, rightly understood, gives us much more than a technique for computing certain scientific problems; it increases our power of conception to an astounding degree, and liberates us from the concrete, visual, and tactual habits of thought that bind the naïve mind so closely to Mother Earth. The intellectual life of man moves between the two extremes of *concrete experience* and *abstract thought*. Near the lower limit lie those intelligent functions which we share, presumably, with the higher animals—our awareness of objects, sensuous memory, expectation, the recognition of simple means to ends. Near the upper limit we may place such intellectual constructions as pure mathematics and symbolic logic, which are entirely formal. The mental life of a generally educated individual ought to run on a plane somewhat above the mean between these two types of thinking. He should be capable of all kinds of ideation. A cultured mind is not one that is stuffed with information, but one that is supple, adaptable to various modes of thought, trained both in sensuous discrimination and in discursive reasoning; a mind that is not bound by any one set of habits, but can shift gears according to its load. Our usual, practical thinking concerns specific situations, concrete experiences, and is weighted with a vast amount of detail; but abstract thinking is unhampered by any irrelevant detail, it is highly selective, and concerns only a few essential aspects of any given experience. It can apply to any number of cases, just because it does not attempt to exhaust any one of them. Therefore, it is vast in its scope and swift in its passage from premise to consequence. It strips its subject matter of all "asides" and individual peculiarities, and thus proceeds from its few premises to their conclusions with a speed that is impossible to a more elaborate process of visualizing concrete situations, which has to stop for all the details of an actual case in order to reason to another actual case. The uneducated mind is geared very low; its progress is ponderous. It pulls hard, without acquiring any momentum. It is a commercial vehicle, which does not move except for some practical purpose; it never runs for pleasure. That is because it always carries its whole cargo of sense-experience, all its known *cases* for contemplation. It does not bring them into the elegant small compass of a *generalization*. Mathematical thought, on the other hand, is concerned entirely with general propositions; its truths are equally valid for any set of things to which you wish to apply them. It is free from all special conditions, for it is a science of things in general. Carlyle, in *Sartor Resartus*, gives one of his characters the humorous title of "Professor of Things in General." He probably was not aware of the implication that the learned Teufeldrökh was Professor of Mathematics.

The first generalization we learn in school is that from three apples, three pencils, three boys, three girls, etc., to the number three. Most children have no difficulty with this amount of formalization; the arithmetical level of thought seems to be the natural level of civilized man. We learn to count and figure as part of our normal mental development. On this plane lies practically all of our thinking—our calculation of money, days, years, measurable goods, physical masses, frequencies, and stresses. Arithmetic is the generalization of concrete facts. A number stands for any group with a given membership. "Three" stands for all triadic collections, "two" for all couples (some mathematicians, as for instance Bertrand Russell, define the number two as "the class of all couples"). Instances of numbers and arithmetical relations may be sought in the world of concrete experience.

The educational value of algebra lies in the fact that it is a definite new step toward abstract conception. Algebraic thinking lies just above the arithmetical level. For *algebra is the generalization of arithmetic.* Its entities stand for numbers. Just as 3 means "any triad," so a means "any number." It does *not* mean some particular number. What is true of a, in the absence of further specifications, is true of every number. Thus, algebra is a generalization of a higher order than arithmetic: its terms are variables whose values are not groups of things, as the values of numbers are, but whose values are numbers. You can readily see that it contributes a very special exercise to the development of the mind. This, as I have pointed out before, is its sole significance in the secondary school curriculum, the only purpose for which it should be taught to the thousands of young people who are never going to use its formulae in solving problems. It must give them the power of abstract conception, of generalization beyond the practical limits of images and other concrete illustrations. One cannot picture the exact meaning of a as one can picture the meaning of 3 by a collection of just three things; for we can count three by any set of counters, but we cannot count a. We must know it purely by definition, i.e., we must *con*ceive the meaning of a or b because it cannot be *per*ceived. No other school subject is designed to expand our mental powers in this direction; the whole task of training the discursive reason, which lifts man above even the cleverest of other creatures, devolves upon the teacher of algebra.

A person who has never taught high school algebra may be unable to estimate the difficulties of the undertaking; but a teacher of philosophy in college is in a particularly good position to estimate the results—the results of school algebra in terms of labor in school. Philosophy does not require the use of algebraic formulae, but it does demand a certain ability to reason *in abstracto*, to conceive of things in general, to appreciate formal relations. This ability seems to be no more developed in high school graduates than in uneducated children. Yet they have all passed at least one year's work in algebra, and most of them have successfully accomplished two. What have they been taught?

Why, they have been taught a great deal of algebra! Most of them know ten times as much as they will ever use. But they have not been taught very much *about* algebra, which is the knowledge they could and would use for humanistic ends. If they have ever heard of generalization, they have long since forgotten it again; most textbooks mention it on page 1 and never refer to it after that. The course hurries on to rules of procedure—rules for adding and subtracting, removing parentheses, squaring, solving for x and y—innumerable rules of the game. The children become as proficient at algebra as their mothers at bridge. As the strange occupation grows more and more familiar, they cease to wonder at the use of letters for numbers, or to puzzle their young heads about the fact that two negatives multiplied together yield a positive product. It is a peculiar fact that a proposition which at first glance looks weird or even nonsensical will seem quite reasonable once it is very familiar. The famous (or infamous) proposition that the earth is round seemed preposterous when it was new, even to many people to whom it had been duly demonstrated; now, because we are used to it, its weirdness has disappeared, and we can believe it without great intellectual effort and even without demonstration. In the same way, the student's original curiosity about the logic of algebra simply dies, even without being satisfied, once he has become familiar enough with the technical practice.

The mastery of algebra—even elementary algebra—requires two intellectual functions, both of which can be developed only by constant practice. They are (1) abstraction, or the recognition of general properties and relationships, and (2) manipulation, or deductive reasoning. The tendency of our educational methods is to develop the latter exclusively. This is unfortunate in so far as there are various other subjects in the curriculum which train the mind in the art of deduction (for instance, arithmetic, geometry, physics), but none that initiates it into the difficulties of abstraction. The result of this one-sided interest is that a student who has learned algebra has acquired a new trick, but not a new insight—the content of his mind has been increased, but not its powers.

What is the reason for the cavalier treatment usually accorded to those vital problems in algebra, namely its meaning and its derivation from arithmetic? The reason for our attitude lies deep in our educational scheme, in what I believe to be a wrong academic ideal: the ideal, or perhaps the fetish, of successful performance. Not only in algebra, but in history and literature as well, our educational methods do not aim at more understanding, but at certain desired responses. Such correct responses can often be more readily procured by sheer habit-training than by any appeal to the conceptual powers. If we teach the student to operate with letters in place of numbers, and can once push him over his natural bewilderment at being asked to multiply the last letter of the alphabet by the first, he can learn, like a robot, to operate with these meaningless symbols as though he were getting sums and products. In fact, he can learn the whole mechanics of the system much faster if he is not constantly called upon to explain the sense of his procedure. Generalization is a difficult art to the naïve, picture-loving mind; whether we can perform it or not makes no difference in the speed of our technique. An efficient teacher, therefore, cannot waste much classroom time on the derivation of $a + b$ from $1 + 2$ and $2 + 3$ and $3 + 4$. She expounds the matter and lets it rest. For she must get through quadratics! The colleges demand it. They do not set examination questions to test the understanding, they offer only problems of technique: to find this cube root, to expand that expression, to determine how long it takes a hen and a half to lay an egg and a half. That, as I have said, expresses a tendency of our age. We are interested in examples rather than general truths, in facts rather than principles, because our ideals are pragmatic. Activity, output, performance are our measures of a man. His behavior, not his thought, is the object of education. (It is not an accident that our generation of psychologists has invented behaviorism.) Therefore, the colleges examine only how much algebra he can "do," not how much he can conceive. But—the more technical dexterity the colleges require, the less time is there for the high school teacher to inculcate fundamental ideas of logic—notions of the general and the particular, of variable and value, relations, and the importance of equivalent forms. These are the mathematical concepts that mark the advance of algebraic thinking over the arithmetical level, on which the mind can always resort to intuition—to images or concrete instances. These concepts are the new acquisitions we are supposed to gain from our traffic with a's and b's; their apprehension shows an actual increase in mental maturity, a step toward scientific and philosophical thought. But having introduced the study of algebra for the sake of its generality and abstractness, we now hush up that aspect of it because by so doing we can teach more algebra! Certainly there is a joker in that game.

The most obvious question to be raised at this point is whether children in high school are able to conceive abstractions. My answer is, no; but they are capable of learning to do so. Most children have very little conceptual power, and it is a sad fact that as adults they are not apt to have much more. This indicates that our education does not train our ability to form a new conception. It leaves the mind as immature and impotent as it was in the untutored state. Instead of trying to stimulate the growth of abstract thought, we arrange our educational methods so as to obviate the need of it, and in this way allow the student's intellectual strength to deteriorate more and more through disuse. Now, instead of avoiding abstract ideas, we should set ourselves the avowed task of introducing them—consciously, clearly, one at a time. There is no reason why a student of average intelligence cannot understand, for instance, that whereas arithmetic expresses relations among particular numbers, such as that $5 + 4 = 9$, algebra is concerned with the relations that hold among any numbers whatever, such as that $4 \times 5 = 5 \times 4$, $20 \times 4 = 4 \times 20$, $6 \times 7 = 7 \times 6$, or generally: any two numbers, call them a and b, if multiplied together, may be taken in either order; in other words, $a \times b = b \times a$. Or consider these propositions: $2 + 3 = 5$; $4 + 6 = 10$; $13 + 16 = 29$; any two numbers added together have a sum that is a number. Any two numbers, call them a and b, have a third number, let us call it c, as their sum; that is, $a + b = c$; or the sum of two numbers is always a number. (Later, when the meaning of positive and negative integers has been introduced, the fact that $5 - 5 = 0$ is equivalent to $5 + (-5) = 0$, or: $a + (-a) = 0$, may be adduced to explain why mathematicians say that 0 is a number.)

If we were not in too great a hurry to get to mechanical devices in algebra and "do examples," any child of high school mentality could learn the principles of correct generalization. Long before the student is asked to translate such expressions as: $a + b + c - 2d$, where $a = 2$, $b = 4$, $c = 5$, $d = 3$, i.e., where definite values are assigned, he should have grasped the significance of: if $a - b = 0$, then $a = b$ (which should later be resorted to again and again to justify the rule of changing signs). These and other fundamental properties of numbers should be brought fully and explicitly to his consciousness because they are simple enough for his understanding and yet perfectly general; in short, they are generalizations which he can perform. He should, moreover, be asked to express algebraically what is common to such a list of equations as this: $5+0 = 5$, $6 + 0 = 6$, $3 + 0 = 3$, $1020 + 0 = 1020$. All his textbook materials and classroom practice aim only in the other direction—namely at finding values for given variables. Thus, he is led to believe that the most important characteristic of algebra is the presence of an "unknown." Never does it occur to him that the sums and products of arithmetic are "unknowns," and that the elementary exercises: $2 + 13 = ?$ $4 \times 5 = ?$ might as well be written $2 + 13 = x$. $4 \times 5 = x$. In each case, find x. The presence of an "unknown quantity" does not take us from arithmetic to algebra; all our arithmetical computations involve the discovery of at least one such unknown term.

There is nothing algebraic about $2 + 13 = x$, or about $2 + a = 10$. Here x and a, respectively, are written for *specific numbers*, not for numbers in general.

The details of teaching algebra from an intellectual, rather than a technical, point of view do not concern us just now; I have merely suggested a few possibilities to show you what sort of stimulation algebra might give to the logical interests and powers of a young student. We want to teach him to recognize algebra as a generalization of arithmetic, to appreciate the relation of a type-form to an example. M1st of our students cannot even tell us why exercises in mathematics are called "examples." They have probably been told, but have been allowed to forget it again. Throughout the course, they should be given some purely arithmetical examples of algebraic forms. It might even be well, at the very beginning, to avoid such mixed forms as $(a + 3)$ $(b - 5)$. A beginner can hardly escape the notion that a and b here stand for specific numbers which are not revealed. If general numbers do occur in connection with specific ones, his attention should be called to the fact that $a + 3$ means "any number plus three" $b - 5$ "any number minus five." (Bertrand Russell, in one of his facetious moments, defined mathematics as "the science in which we never know what we are talking about.")

Having told the class, not once, but repeatedly and insistently, that a, b, and c mean any *three numbers*, equal or unequal to each other, great or small, we come to the next algebraic concept, that of *relations among numbers*. The structure of the number system should here be pointed out—namely that every number has a successor, or next neighbor on the right, and every number except 0 has a predecessor, or next neighbor on the left. Thus the number series is infinite in one direction. The children will readily see that, since there is no upper limit to this series, so that it is quite safe to say quite generally that for every a and b there is a c such that $a + b = c$; but that the same is not true when we *subtract*, or work in the other direction, for here the outfit of available numbers is limited. There are no numbers below 0. We cannot say that any number may be subtracted from any other number and the result will always be a number; that $a - b = c$ always. Here our generalization of arithmetic seems to break down; $a - b = c$ is not generally true as long as 0 cuts off one end of the number series.

Here we arrive quite logically at the negative numbers. The thermometer, which may drop below 0, is a familiar household example of purely relative number. By means of the relative numbers, we may take away and take away without specifying any limit such as 0. We simply go into debt; the result of a subtraction may take us below 0. Thus, the general proposition, for every a and b there is a c such that $a - b = c$, has been saved. (I have no doubt that in using the expressions $a + b = c$ and $a - b = c$ one must train the student—perhaps with considerable pains—to realize that c is *not* a particular number, so that we are *not* asserting that the sum of two numbers is the same as their difference.)

This matter of negative numbers is well explained in most of our textbooks, but often a good deal of confusion remains in the student's mind regarding the signs $+$ and $-$; for $+$ may indicate the place of a term in the number series, meaning that it is above 0, or it may be a sign of addition. Here one must tell the class explicitly that $+$ is used for two purposes and has two distinct, though related, meanings: $+$ *belonging to a number* puts it to the right, or as we say "above," the number 0, and $+$ *between two terms* indicates a way of putting them together; in the old days when all numbers except 0 were above 0, this operation always took us further in the plus direction, wherefore the sign of this operation came to be used also as the above-zero sign, and the sign which took us toward 0 became the below-zero sign for relative numbers. Both signs at present have double meanings. Unless these are clearly understood, the student cannot be expected to comprehend as well as believe that $a - (-b) = a + b$; and in mathematics we should be trained to comprehend everything that we believe.

The fact that numbers stand in definite relations to each other not only makes it possible to perform operations, i.e., to determine any number as so-and-so far from any other number, but it also enables us to express the same number in many different ways; to write a as the number which is so-and-so far above b, or below c, or lying evenly between any number and its negative, as 0 does. We may describe 2 as the successor of 1, i.e., as $1 + 1$; or as the predecessor of 3, i.e., as $3 - 1$; or as the number that must be added to 4 in order to reach 6, i.e., $6 - 4$. This takes us to the third great principle of reasoning, which we *use* in all mathematics, and should recognize explicitly through the generalized formulae of algebra: *the principle of equivalent forms.* Why do all the examples at which we labor in school take the form of equations? Because the recognition of equivalent forms is the whole art of deduction. This art can be learned only by patient practice, and this the standard course is designed to give; for at this point our algebra course in school begins. But all too often the practice is performed purely as a chore, without logical insight, and the importance of the principle is not understood at all.

The principle of equivalent forms is the basis not only of mathematical deduction, but of all scientific analysis. Water $= H_2O$; we cannot relate water to other chemical structures until we think of it under this special form, as a compound of certain elements. Likewise, we cannot measure a velocity without conceiving it as equivalent to a certain relation between mass, distance, and time. And not only in physics, but also in metaphysics, what are we really doing but seeking some form under which the various items of experience may be seen in relation to one another? Their obvious forms do not allow this; but where one way of seeing them baffles us, another may show without difficulty how they fit into one system. Even in everyday life, we are constantly resorting to various forms of the same thing; it would be very hard to buy small articles at absurd fractions of a dollar, if we could not treat the dollar under its equivalent form of one hundred cents. Ingenuity in thinking—whether in practical, scientific, or philosophical thinking—is primarily the art of recognizing widely different things as different *forms* that amount to the same thing—as ice, vapor, and water are forms of the same thing, or equivalents—or else discovering new ways in which a familiar thing may be treated so as to reveal some hitherto unknown relation; as lightning, for instance, could not be fitted into a system of physics until it was thought of as an electrical discharge. Algebra offers great possibilities for ingenuity; type-forms of equivalences should not merely be learned and applied, but, whenever they are simple enough, they should be derived or analyzed in class.

To present a detailed program for the logical study of algebra is beyond my ambitions; that is a task for experienced teachers and educational theorists. I am merely pointing out a few of the fundamental ideas which mathematics—algebra in particular—should contribute to the growth of the human mind. Of course, I realize that any course based upon these requirements would have to proceed much more slowly than the standard course. There would be lessons spent on finding examples of the order of relative number—the thermometer, credits and debits, right and left, before and after (though it is a peculiarity of our chronology that, being invented and adopted before the introduction of 0, it has no year 0, and thus does not conform to the pattern of the algebraic number series. A modern calendar would certainly denote the year of the Nativity as 0). There would be questions and discussions of the meaning of signs, of the reasons why $a + b$ is an expression whereas $a \times b$ is a term, what it means to multiply by a negative number, i.e., apparently, to take a term a negative number of times, and the meaning of "constant" and "variable." Some of these problems—not merely their answers, but the problems themselves—would have to be carefully circumvented at the beginning, so as not to swamp and bewilder the blundering, uninitiated minds. The course would move very slowly at first; but it would make algebra appear as a study, not a game. Examples would become examples *of* something, and formulae would be summaries, not rules.

Whether algebra can be taught on these principles to the average adolescent mind, I do not not know; the scheme may be Utopian. Only careful pedagogical experimenting can decide such a question. But if algebra cannot be imparted in this or some similar way to everybody, then it should not be a required subject in a general arts course. In the technical schools, where ability in computation is the chief aim of algebra, the present method of presentation is probably by far the best; but in the classical course it is out of place. There algebra should be taught from the logical standpoint to those who can understand it, and not at all to those who cannot. It should be elective, like Greek. I do not mean by this that it should be recommended only to children of marked mathematical talent; on the contrary, all children with a superior intelligence quotient should be urged to study it, the practical, concrete-minded ones as well as those who are naturally abstract thinkers; for a lack of natural bent is no reason for neglecting a faculty which normally requires some education. We do not advise awkward children to avoid dancing and gymnastics, but seek rather to help them over their ineptitudes. Any mind which is generally active and keen should be developed as far as possible in several directions. The child whose thinking is preponderantly verbal should have generous training in sensuous discrimination and visual judgment so as not to let his lesser talents deteriorate; there is, of course, no sense in making him use visual methods where his own are natural and suitable—that would be like reforming left-handedness, an interference with nature—but he should be given plenty of tasks that absolutely require visual attention. Likewise, a child whose mind is above the average, but whose thinking is predominantly picturesque and particular, should be systematically taught the art of generalization, lest his logical powers be lost completely through the unconscious habit of evading logical tasks.

The average individual does not take kindly to abstract thought; this weakness has been recognized by our contemporary writers and educators, and in their anxiety to impart huge doses of information to thousands of average people they have worked out a remarkable technique of avoiding abstractions. Thus it has become a fashion and almost a mania to present difficult logical matters—relativity, quantum theory, Gestalt theory, or what-not—in all sorts of enticing imagery. This popular practice has the pernicious consequence of leading people to suppose that they understand anything which they can associate with an interesting mental picture or of which they have been shown an example. Naturally, if they believe that the thoughts of Einstein can be put at their command through simple language and concrete illustration, they will regard difficult mathematical abstractions as gratuitous and silly. Thus our own constant catering to the preferences of the untutored mind has created a sort of horror and suspicion of abstractions which stands in the way of many a student's natural interest and pleasure in algebra or logic. If mathematics is ever to be a real contribution to the intellectual development of your pupils, you must combat this attitude by your own attitude—by emphasizing the intrinsic interest of a general truth quite as much as the usefulness of certain computations, by recognizing the intellectual virtues in algebra—its superiority to arithmetic in point of generality, its perfect logic, its relation to philosophy. If we teach algebra as part of a humanistic course, it must be because we hold an academic ideal of easy, correct, and unrestricted thinking. A trained mind likes to think. An unskilled mind is afraid of the chore. If the high school is to have any deep effect on the lives of its students, it must contribute not only to their stock of subject matter for thinking, but also to their powers of thought. Of all school disciplines, algebra is the special training ground of reason. It offers the first logical technique, requires the first sophistication of reason, and is the ante-chamber of science and philosophy.

Mathematics in colleges and universities

A newly released report concerning mathematics education is the most comprehensive depth study of programs within a discipline ever undertaken in the United States. Both undergraduate and graduate programs were surveyed in the study, and information was solicited on curriculums, degrees, course offerings, enrollments, credit requirements, examination requirements, special features, innovations, and trends.

A few of the many topics are:

- Credit-hour requirements for the mathematics teaching curriculum
- Kinds of degrees awarded in the mathematics teaching curriculum
- Extent of prerequisite instruction and provision for credit
- Placement examinations in mathematics
- Master's programs in mathematics
- Master's programs specially designed for the teaching of mathematics
- Institutions at which doctoral degrees specially designed for the teaching of mathematics may be earned through evening and/or Saturday study, or through summer study
- The undergraduate mathematics club

The study was conducted by Clarence B. Lindquist, specialist in mathematics and physical sciences, U.S. Office of Education. The 104-page report of the study, *Mathematics in Colleges and Universities*, has just been published and is available for 60¢ from the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 20402.